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Remarks on Quantization of Classical r-Matrices

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Abstract

If a classical r-matrix r is skewsymmetric, its quantization R can lose the skewsymmetry property. Even when R is skewsymmetric, it may not be unique.

Let r be a classical r-matrix. In general, it means that we have a family of vector spaces $\{V_\alpha\}$, $\alpha \in \mathcal{A}$, and a collection of linear operators

$$r(\alpha,\beta):\ V_\alpha \otimes V_\beta \to V_\beta \otimes V_\alpha, \qquad \forall\ \alpha \neq \beta \in \mathcal{A}, \tag{1}$$

satisfying the misnamed "Classical Yang-Baxter" equation (CYB)

$$\begin{aligned}[c(r)]^{\varphi\psi\xi}_{ijk}(\alpha,\beta,\gamma) := \Big(r(\alpha,\beta)^{s\varphi}_{ij} r(\beta,\gamma)^{\xi\psi}_{sk} + r(\alpha,\beta)^{\psi s}_{ij} r(\alpha,\gamma)^{\xi\varphi}_{sk}\Big)\\ + c.p.(i,j,k;\varphi,\psi,\xi;\alpha,\beta,\gamma) = 0\end{aligned} \tag{2}$$

where "$c.p$" stands for the sum on cyclically permuted triples of indices indicated, and $r(\alpha,\beta)^{uv}_{ij}$ are the matrix elements of the operators $r(\alpha,\beta)$ (1) in a collection of fixed basises:

$$r(\alpha,\beta)\left(e^\alpha_i \otimes e^\beta_j\right) = r(\alpha,\beta)^{\ell k}_{ij} e^\beta_\ell \otimes e^\alpha_k; \tag{3}$$

the convention of summation over repeated upper-lower indices is in force.

In most applications, all the vector spaces V_α are isomorphic to each other, $V_\alpha \approx V$; in addition, often, – but not always, – the operator $r: V\otimes V \to V \otimes V$ is skewsymmetric:

$$PrP = -r, \qquad r^{k\ell}_{ij} = -r^{\ell k}_{ji}, \tag{4}$$

where P is the permutation operator,

$$P(x\otimes y) = y \otimes x. \tag{5}$$

We shall consider this particular framework from now on.

To quantize a given r-matrix r is to find an operator family

$$R = R(h): \ V \otimes V \to V \otimes V, \tag{6}$$

depending upon a parameter h, such that

$$R(h) = P + hr + O(h^2), \tag{7}$$

and R satisfies the Artin braid relation (also misnamed as the "Quantum Yang-Baxter" equation, QYB):

$$R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23}, \tag{8}$$

where this equality of operators acting on $V \otimes V \otimes V$ employs the standard notation

$$R^{12}(x \otimes y \otimes z) = R(x \otimes y) \otimes z, \qquad R^{23}(x \otimes y \otimes z) = x \otimes R(y \otimes z). \tag{9}$$

How does the skewsymmetry condition on r, (4), translate into $R = R(h)$?

There are at least two possible, *logically independent*, answers, only one of which is correct.

The first one is what is commonly accepted in the literature under the name of "unitarity":

$$R(h)^{-1} = R(h), \tag{10a}$$

or

$$R(q)^{-1} = R(q), \tag{10b}$$

in the multiplicative notation $q = e^h$.

The second one I shall call, for want of a better term, the mirror symmetry:

$$R^{\mathcal{M}}(h) = R(-h). \tag{11}$$

Here $R^{\mathcal{M}}$ is the operator acting as the mirror image of R. If

$$R\left(e_i \otimes e'_j\right) = R^{k\ell}_{ij} e'_k \otimes e_\ell, \tag{12}$$

then

$$R^{\mathcal{M}}\left(R^{k\ell}_{ij} e_\ell \otimes e'_\kappa\right) = e'_j \otimes e_i. \tag{13}$$

This definition, useful as it is, is *not* connected to skewsymmetry of r.

The classical r-matrix r appears as the h^1-term in the h-expansion of the Quantum R-matrix $R(h)$ around $h = 0$. The terms in h of orders higher than 1 recede away in the quasiclassical passage. The examples that follow demonstrate that these higher-order terms can have distinctly anti-Prussian character and break out strict orders and symmetries. (In [1] Drinfel'd proved that every skewsymmetric classical r-matrix r represents h^1-part of some skewsymmetric Quantum $R-$matrix R. The question of additional parameters in R was not addressed there, or elsewhere.)

In the 1^{st} example, $\dim(V) = 2$ and the R-matrix $R = R(h;\theta)$ acts on $V \otimes V$ (in a chosen basis) as

$$R\left(e_0 \otimes e'_0\right) = e'_0 \otimes e_0, \tag{14}$$

$$R\left(e_0 \otimes e_1'\right) = \left(e_1' + he_0'\right) \otimes e_0, \tag{15}$$

$$R\left(e_1 \otimes e_0'\right) = e_0' \otimes (e_1 - he_0), \tag{16}$$

$$R\left(e_1 \otimes e_1'\right) = e_1' \otimes e_1 + \theta h^2 e_0' \otimes e_0. \tag{17}$$

Here θ is an arbitrary constant. The Artin relation (8) is easily verified. The h^1-terms comprise the r-matrix

$$r_{ij}^{k\ell} = \delta_0^k \delta_0^\ell \left(\delta_{ij}^{01} - \delta_{ij}^{10}\right) \tag{18}$$

which is obviously skewsymmetric. The R-matrix $R(h;\theta)$ is, however, not unitary unless $\theta = 0$. Also, it's easy to see that

$$R^{\mathcal{M}}(h;\theta) = R(-h;-\theta), \tag{19}$$

so that this R-matrix is not mirror-symmetric either, again unless $\theta = 0$.

Our 2^{nd} example is a little bit more elaborate, with $\dim(V) = 3$. Here the R-matrix *is* both skewsymmetric and mirror-symmetric, but it depends upon one extra parameter, in addition to the quantization parameter h, thus exhibiting clearly nonuniqueness of quantization of classical r-matrices.

Fixing a basis (e_0, e_1, e_2) in V, we set

$$R\left(e_0 \otimes e_0'\right) = e_0' \otimes e_0, \tag{20.1}$$

$$R\left(e_0 \otimes e_1'\right) = \left(e_1' + he_0'\right) \otimes e_0, \tag{20.2}$$

$$R\left(e_1 \otimes e_0'\right) = e_0' \otimes (e_1 - he_0), \tag{20.3}$$

$$R\left(e_1 \otimes e_1'\right) = e_1' \otimes e_1; \tag{20.4}$$

$$R\left(e_0 \otimes e_2'\right) = \left(e_2' + he_1' + \frac{h^2}{2}e_0'\right) \otimes e_0, \tag{21.1}$$

$$R\left(e_2 \otimes e_0'\right) = e_0' \otimes \left(e_2 - he_1 + \frac{h^2}{2}e_0\right), \tag{21.2}$$

$$R\left(e_1 \otimes e_2'\right) = e_2' \otimes (e_1 + he_0) + h^2\left(\frac{1}{2}e_1' + \lambda he_0'\right) \otimes e_0, \tag{21.3}$$

$$R\left(e_2 \otimes e_1'\right) = \left(e_1' - he_0'\right) \otimes e_2 + h^2 e_0' \otimes \left(\frac{1}{2}e_1 - \lambda he_0\right), \tag{21.4}$$

$$\begin{aligned} R\left(e_2 \otimes e_2'\right) = {} & e_2' \otimes \left(e_2 + he_1 + \frac{h^2}{2}e_0\right) \\ & - he_1' \otimes \left(e_2 + \tilde{\lambda}h^2 e_0\right) + h^2 e_0' \otimes \left(\frac{1}{2}e_2 + \tilde{\lambda}he_1\right). \end{aligned} \tag{21.5}$$

Here λ is the new free parameter, and

$$\tilde{\lambda} = \lambda - \frac{1}{4}. \tag{22}$$

From formulae (20) we see that the previous example (14)–(17) is embedded into this one, with $\theta = 0$. It's immediate to check that

$$R(h;\lambda)^2 = \mathbf{1}, \tag{23}$$

$$R^{\mathcal{M}}(h;\lambda) = R(-h;\lambda), \tag{24}$$

so that our R-matrix is both skewsymmetric and mirror-symmetric. Also, the h^1-part of $R(h;\lambda)$ is given by the flag-type formula

$$r_{ij}^{k\ell} = (i-c)\delta_i^\ell \delta_{j-1}^k - (j-c)\delta_{i-1}^\ell \delta_j^k, \qquad 0 \le i,j,k,\ell \le \dim(V) - 1, \tag{25}$$

where c is an arbitrary constant. [In our case $c = 1$, but this constant can be adjusted to any desired value by an appropriate nonlinear transformation; in particular, we can make

$$c = \frac{\dim(V) - 1}{2} \tag{26}$$

to have the determinant in $GL(V)$ being central in the induced Lie-Poisson structure [2]. In this language, the R-matrix (20)–(21) defines the Quantum Group $\mathrm{Mat}_{h;\lambda}(3)$, a 3-dimensional analog of the 2-dimensional Quantum Group $\mathrm{Mat}_h(2)$.] The checking of the Artin relation for the R-matrix (20)–(21) is easy but tedious; the mirror property (24) cuts the verification procedure by 1/3; there are still more symmetries present in this R-matrix which will allow another 1/3 of the checking labor to be avoided.

How many additional constants should one expect when quantizing a skewsymmetric classical r-matrix and requiring the Quantum R-matrix to be skewsymmetric and mirror-symmetric? For the case of the r-matrix (25), I expect the total number of additional parameters (the λ's) to be

$$\dim(V) - 2, \tag{27}$$

and in general it probably could never be larger no matter what r is; dropping off the mirror-symmetry condition increases the number of possible parameters by 1.

References

[1] Drinfel'd V.G., On Constant Quasiclassical Solutions of Quantum Yang-Baxter Equation, *Sov. Math. Dokl.*, 1983, V.28, 667–671.

[2] Kupershmidt B.A., Poisson Relations Between Minors and their Consequences, *J. Phys. A*, 1994, V.27, L507–L513.